Extremal CFTs, The Pure Quantum Gravity and Conformal Bootstrap

Jin-Beom Bae

Korea Institute for Advanced Study

Based on ArXiv:1610.05814
work with Sungjay Lee and Kimyeong Lee

Current Topics in String Theory
2016. 12. 16

- General Aspects of Conformal Field Theory
  - The conformal group $SO(D,2)$ have finite number of generators : $P_\mu(translation), M_{\mu\nu}(rotation), D(dilatation)$ and K_μ $(special\ conformal)$
  - Every quantum states in CFTs are labeled by $(\Delta,\ \ell)$
  - Unitary bound is given by

$$\Delta \geq \frac{D}{2} - 1 \text{ (for spin 0)}, \quad \Delta \geq D + \ell - 2 \text{ (for spin } \ell)$$

  - The critical exponents η and ν are given by $\Delta_\phi = \frac{D}{2} - 1 + \frac{\eta}{2}$ and $\Delta_{\phi^2} = D - \frac{1}{\nu}$.
- Two-dimensional CFT
  - The conformal algebra $SO(D,2)$ is extended to the Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \quad n, m \in \mathbb{Z}$$

  - 'Solvable' CFTs : The data of unitary minimal models$(p' = p+1)$:

$$c(p) = 1 - \frac{6}{p(p+1)}, \quad h_{r,s}(p) = \frac{((p+1)r - ps)^2 - 1}{4p(p+1)}$$

  - $p = 3$: Ising, $p = 4$: tricritical Ising, $p = 5$: three-state Potts model, $\cdots$

- Four-point function from conformal covariance
  - Conformal symmetry fix the structure of the correlation functions.

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{-2\Delta_\phi}}$$

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{f_{ijk}}{|x_{12}|^{\Delta_{\phi_i}+\Delta_{\phi_j}-\Delta_{\phi_k}}|x_{23}|^{\Delta_{\phi_j}+\Delta_{\phi_k}-\Delta_{\phi_i}}|x_{31}|^{\Delta_{\phi_k}+\Delta_{\phi_i}-\Delta_{\phi_j}}}$$

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \frac{1}{|x_{12}|^{\Delta_i+\Delta_j}|x_{34}|^{\Delta_k+\Delta_l}}\left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{13}} G\left(u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\right)$$

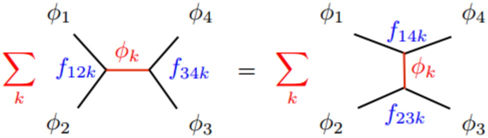
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \to \frac{1}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} G(u, v)$$

  - Four-point correlation function have two DOF (u, v), for $D \geq 2$.
- Four-point funtion from operator product expansion
  - Applying 12/34 channel OPE twice,

$$\langle \overbrace{\phi(x_1)\phi(x_2)}\overbrace{\phi(x_3)\phi(x_4)} \rangle = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 \frac{g_{\Delta,l}(u, v)}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}} = \frac{G(u, v)}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}}, \qquad G(u, v) = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 g_{\Delta,l}(u, v)$$

Function $g_{\Delta,\ell}(u, v)$ is called by *conformal block*.

- The idea of conformal bootstrap program [Rattazzi, Rychkov, Tonni, Vichi 08]
  - Unitarity : $\Delta \geq \frac{D}{2} - 1$ for spin 0 field, $\Delta \geq D + \ell - 2$ for spin ℓ field.
    $f^2_{\phi\phi\mathcal{O}} > 0$.
  - Crossing symmetry of the 4-point correlation function



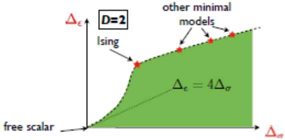
  - The *Bootstrap Equation* from Unitarity and Crossing symmetry :

$$\boxed{u^{\Delta} - v^{\Delta} = \sum_{\substack{\Delta,\ell \\ \Delta \geq \Delta_{unit}}} f^2_{\phi\phi\mathcal{O}} \left(v^{\Delta} g_{\Delta,\ell}(u,v) - u^{\Delta} g_{\Delta,\ell}(v,u) \right)}$$

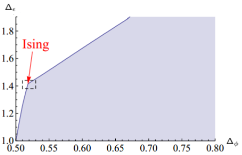
  - The conformal block $g_{\Delta,\ell}(u,v)$ is determined from the conformal Casimir equation. (cf) Zamolodchikov recursive relation.) [Kos, Simmons-Duffin, Poland 13]
  - Apply linear functional to the bootstrap equation $\longrightarrow$ Semi-definite Programming
    [Simmons-Duffin, Poland 11]

Two($L_0, L_{\pm 1}$)- and Three-dimensional Numerical Results

- The numerical results of two-dimensional (Left) and three-dimensional (Right) conformal bootstrap are given by,



[Rychkov 11]



[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi 12]

- The *kink* appears on the numerical bounds. One can read its coordinate $(\Delta_\phi, \Delta_{\phi^2})$, they indeed agree with the result of ϵ-expansion $\oplus$ Borel resummation!
- For comparison(3D), [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi 14]

Critical exponent	ε expansion(1998)	Monte-Carlo(2010)	Bootstrap(2014)
η	0.03650(500)	0.03627(10)	0.03631(3)
ν	0.63050(250)	0.63002(10)	0.62999(5)

Some Questions

- Numerical bootstrap with Virasoro algebra?
- Three-dimensional version of minimal theory?
- AdS/CFT for tiny cosmological constant. Large gapped CFT?

- State counting of Virasoro descendants
  - The Virasoro descendants are generated by L_{-n} and $\bar{L}_{-n}(n \geq 2)$).
  - Start from the Vacuum, we have,

$$|\Omega\rangle,\ L_{-2}|\Omega\rangle,\ L_{-3}|\Omega\rangle,\ {\color{red}L_{-2}^2|\Omega\rangle,\ L_{-4}|\Omega\rangle},\ {\color{blue}L_{-2}L_{-3}|\Omega\rangle,\ L_{-5}|\Omega\rangle},\cdots.$$

  - Their contributions are counted by

$$\mathcal{Z}_{vac} = \Big|\underbrace{q^{-\frac{c}{24}}\frac{(1-q)}{\eta(\tau)}}_{\chi_{1,1}(\tau)}\Big|^2 = |q^{-\frac{c}{24}}(1+q^2+q^3+{\color{red}2}q^4+{\color{blue}2}q^5+\cdots)|^2$$

  - To complete modular invariance, *we need additional contributions.*
    For example, the partition function of Ising model ($c=\frac{1}{2}$):

$$\mathcal{Z}_{p=3}(\tau,\bar{\tau}) = \underbrace{|\chi_{1,1}(\tau)|^2}_{I} + \underbrace{|\chi_{1,2}(\tau)|^2}_{\sigma} + \underbrace{|\chi_{2,1}(\tau)|^2}_{\epsilon}$$

    For three-state Potts model($c=\frac{6}{7}$),

$$\mathcal{Z}_{p=5}(\tau,\bar{\tau}) = |\chi_{1,1}(\tau)+\chi_{1,4}(\tau)|^2 + |\chi_{1,2}(\tau)+\chi_{1,3}(\tau)|^2 + 2|\chi_{3,3}(\tau)|^2 + 2|\chi_{3,4}(\tau)|^2$$

- The pure quantum gravity in AdS$_3$
  - The Eintein-Hilbert action of the pure quantum gravity in AdS$_3$ is,

$$\mathcal{I} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \Big(R + \underbrace{\frac{2}{\ell^2}}_{\Lambda<0} \Big).$$

- The Chern-Simons description [Witten 07]
  - The pure gravity in AdS$_3$ is perturbatively equivalent to the two copies of Chern-Simons theory.

$$\mathcal{I}^{CS} = \frac{1}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$\mathcal{I} = k_L \mathcal{I}_L^{CS} + k_R \mathcal{I}_R^{CS} = \frac{k_L + k_R}{2} (\mathcal{I}_L^{CS} - \mathcal{I}_R^{CS}) + (k_L - k_R) \frac{(\mathcal{I}_L^{CS} + \mathcal{I}_R^{CS})}{2}$$

  - Brown-Henneaux computed central charge $c^{BH} = \frac{3\ell}{2G}$ for the case of $k_L - k_R = 0$.
  - Combining with $k_L + k_R = \frac{\ell}{8G}$, $k = k_L = k_R = \frac{\ell}{16G}$. Therefore, $c = c_L = c_R = 24k$.
- Holomorphic Factorization [Witten, Maloney 07]
  - In this work, we assumed the pure gravity in AdS$_3$ admit holomorphic factorization.
  - We will focus one copy of them. (i.e., $h \in \mathbb{Z}$ and $\bar{h} = 0$.)

- The pure quantum gravity in AdS$_3$
  - The Eintein-Hilbert action of the pure quantum gravity in AdS$_3$ is,

$$\mathcal{I} = \frac{1}{16\pi G}\int d^3x\sqrt{g}\Big(R + \underbrace{\frac{2}{\ell^2}}_{\Lambda<0}\Big).$$

- The Chern-Simons description [Witten 07]
  - The pure gravity in AdS$_3$ is perturbatively equivalent to the two copies of Chern-Simons theory.

$$\mathcal{I}^{CS} = \frac{1}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

$$\mathcal{I} = k_L\mathcal{I}_L^{CS} + k_R\mathcal{I}_R^{CS} = \frac{k_L+k_R}{2}\Big(\frac{1}{\pi\ell}\int d^3x\sqrt{g}(R+\frac{2}{\ell^2})\Big) + (k_L-k_R)\frac{(\mathcal{I}_L^{CS}+\mathcal{I}_R^{CS})}{2}$$

  - Brown-Henneaux computed central charge $c^{BH} = \frac{3\ell}{2G}$ for the case of $k_L - k_R = 0$.
  - Combining with $k_L + k_R = \frac{\ell}{8G}$, $k = k_L = k_R = \frac{\ell}{16G}$. Therefore, $c = c_L = c_R = 24k$.
- Holomorphic Factorization [Witten, Maloney 07]
  - In this work, we assumed the pure gravity in AdS$_3$ admit holomorphic factorization.
  - We will focus one copy of them. (i.e., $h \in \mathbb{Z}$ and $\bar{h} = 0$.)

- BTZ blackhole and primary state [Witten 07]
  - The mass, spin and entropy of BTZ black hole(classically) are given by

$$M = \frac{1}{\ell}(L_0 + \bar{L}_0), \quad J = L_0 - \bar{L}_0, \quad S_{L,R} = 4\pi\sqrt{k_{L,R} L_0}$$

  It is natural to take $L_0 \geq 1$ (L_0=0, the classical entropy vanish).
  - Therefore, the full partition function have the form of

$$\mathcal{Z} = q^{-k}\frac{(1-q)}{\eta(\tau)} + \mathcal{O}(q)$$

  $\mathcal{O}(q)$ is completely determined from the Virasoro descendants.
  - They are called by *primary states*, correspond to the BTZ black holes.
  - We call *Extremal Conformal Field Theory* when $h_{l.p} - h_{Vac} = k + 1$.

- $k = 1$ construction [Frenkel, Lepowsky, Meurman 84]
  - 71 CFTs : 70 with Kac-Moody $\oplus$ 1 without Kac-Moody [Schellekens 92]
  - The modular invariant Klein j-function $j = \frac{1728E_4^3}{E_4^3 - E_6^2}$ is the unique partition function

$$\mathcal{Z}_{k=1} = j - 744 = q^{-1} + \underbrace{196884}_{196883+1} q + \underbrace{21493760}_{21296876+196883+1} q^2 + \cdots$$

- Holography of the three-dimensional pure quantum gravity
  - $c \geq 25$: Debates on the large k Extremal CFTs. [Gaberdiel 07] [Gaiotto 08] [Iizuka, Tanaka, Terashima 15]
  - $c \leq 1$: $p > 4$ minimal do not admit the consistent pure quantum gravity. [Castro, Gaberdiel, Hartman, Maloney, Volpato 11]
- Moving to two-dimensional CFT
  - The conformal block is replaced by Virasoro block. [Zamolodchikov 86]

$$\mathcal{F}\left(c, h_\phi, h_\mathcal{O}; z\right) = \left[z(1-z)\right]^{\frac{c-1}{24}-2h_\phi}\left[16\ q(z)\right]^{h_\mathcal{O}-\frac{c-1}{24}}\vartheta_3\left(q(z)\right)^{\frac{c-1}{2}-16h_\phi}H\left(c, h_\phi, h_\mathcal{O}; z\right)$$

The function $H\left(c, h_\phi, h_\mathcal{O}; z\right)$ is obtained by recursive relation.

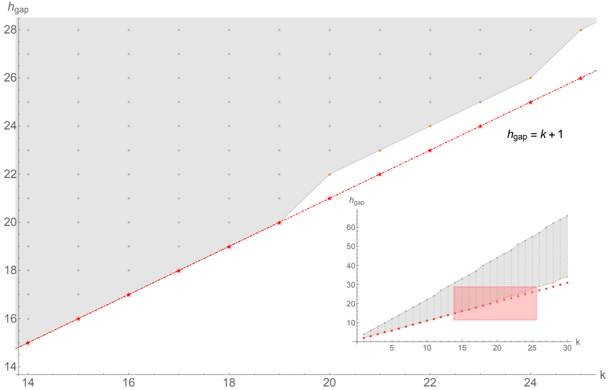
$$H(c, h_\phi, h_\mathcal{O}, q(z)) = 1 + \sum_{m,n\geq 1}\frac{(16\ q(z))^{mn}R_{m,n}(c, h_\phi)}{h_\mathcal{O} - h_{m,n}(c)}H(c, h_\phi, h_{m,n}(c) + mn, q(z)) \ .$$

  - The *Bootstrap Equation* from Unitarity and Crossing symmetry :

$$\boxed{0 = \sum_h f^2_{\phi\phi\mathcal{O}}\left(\mathcal{F}\left(c, h_\phi, h_\mathcal{O}; z\right) - \mathcal{F}\left(c, h_\phi, h_\mathcal{O}; 1-z\right)\right)}$$

  - We examined four-point correlation function of lowest primary operator, namely $h_\phi = k + 1$. $c = 24k$, therefore our input parameters are k and $h_\mathcal{O}$.

- Numerical result of ECFTs conformal bootstrap



- h_{gap} is adjustable parameter. This means conformal weight of the lowest primary. For extremal CFTs, $h_{gap} = k + 1$. (Red line in this plot)
- When $k \geq 20$, the red line **do not included** in the shaded region.
- This suggests ECFTs with $k \geq 20$ cannot be consistent with bootstrap equation.
- Under the assumption of holomorphic factorization, this may suggests non-existence of the pure quantum gravity in AdS_3.

- Conclusion and Outlook

  • The two-dimensional extremal conformal field theories are a candidate for the boundary CFTs of the pure quantum gravity in AdS_3.

  • With the constraint $h_{gap} = k + 1$, the numerical bootstrap suggest no consistent quantum theory at weakly curved limit.

  • Our conclusion is based on the holomorphic factorization.

  • Can we relax the assumption of holomorphic factorization?

  • Supersymmetric extension of extremal CFTs? [Harrison 16]

  • Is there two-dimensional large-gapped CFT? Or, higher dimensional large-gapped CFT? And its relation to the quantum gravity?

Thank you for paying attention!